AMENABLE ACTIONS, INVARIANT MEANS AND BOUNDED COHOMOLOGY

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ABSTRACT. We show that topological amenability of an action of a countable discrete group on a compact space is equivalent to the existence of an invariant mean for the action. We prove also that this is equivalent to vanishing of bounded cohomology for a class of Banach G-modules associated to the action, as well as to vanishing of a specific cohomology class. In the case when the compact space is a point our result reduces to a classic theorem of B.E. Johnson characterising amenability of groups. In the case when the compact space is the Stone-Čech compactification of the group we obtain a cohomological characterisation of exactness for the group, answering a question of Higson.

1. INTRODUCTION

An invariant mean on a countable discrete group G is a positive linear functional on $\ell^\infty(G)$ which is normalised by the requirement that it pairs with the constant function 1 to give 1, and which is fixed by the natural action of G on the space $\ell^\infty(G)^*$. A group is said to be amenable if it admits an invariant mean. The notion of an amenable action of a group on a topological space, studied by Anantharaman-Delaroche and Renault [1], generalises the concept of amenability, and arises naturally in many areas of mathematics. For example, a group acts amenably on a point if and only if it is amenable, while every hyperbolic group acts amenably on its Gromov boundary.

In this paper we introduce the notion of an invariant mean for a topological action and prove that the existence of such a mean characterises amenability of the action. Moreover, we use the existence of the mean to prove vanishing of bounded cohomology of G with coefficients in a suitable class of Banach G modules, and conversely we prove that vanishing of these cohomology groups characterises amenability of the action. This generalises the results of Johnson [6] on bounded cohomology for amenable groups.

Another generalisation of amenability, this time for metric spaces, was given by Yu [10] with the definition of property A. Higson and Roe [7] proved a remarkable result that unifies the two approaches: A finitely generated discrete group G (regarded as a metric space) has Yu's property A if and only if the action of G on its Stone-Čech compactification βG is topologically amenable, and this is true if and only if G acts amenably on any compact space. Ozawa proved [9] that such groups are exact, and indeed property A and exactness are equivalent for countable discrete groups equipped with a proper left-invariant metric.

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To generalise the concept of invariant mean to the context of a topological action, we introduce a Banach G-module $W_0(G,X)$ which is an analogue of $\ell^1(G)$, encoding both the group and the space on which it acts. Taking the dual and double dual of this space we obtain analogues of $\ell^\infty(G)$ and $\ell^\infty(G)^*$. A mean for the action is an element $\mu \in W_0(G,X)^{**}$ satisfying the normalisation condition $\mu(\pi) = 1$, where the element π is a summation operator, corresponding to the pairing of $\ell^1(G)$ with the constant function 1 in $\ell^\infty(G)$. A mean μ is said to be invariant if $\mu(g \cdot \varphi) = \mu(\varphi)$ for every $\varphi \in W_0(G,X)^*$, (Definition 13).

With these notions in place we give the following very natural characterisation of amenable actions.

Theorem A. *Let G be a countable discrete group acting by homeomorphisms on a compact Hausdorff topological space X. The action is amenable if and only if there exists an invariant mean for the action.*

We then turn to the question of a cohomological characterisation of amenable actions. Given an action of a countable discrete group G on a compact space X by homeomorphisms we introduce a submodule $N_0(G,X)$ of $W_0(G,X)$ associated to the action and which is analogous to the submodule $\ell^1_0(G)$ of $\ell^1(G)$ consisting of all functions of sum 0. Indeed when X is a point these modules coincide. We also define a cohomology class $[J]$, called the Johnson class of the action, which lives in the first bounded cohomology group of G with coefficients in the module $N_0(G,X)^{**}$. We have the following theorem.

Theorem B. *Let G be a countable discrete group acting by homeomorphisms on a compact Hausdorff topological space X. Then the following are equivalent*

(1) *The action of G on X is topologically amenable.*

(2) *The class $[J] \in H^1_b(G, N_0(G,X)^{**})$ is trivial.*

(3) $H^p_b(G, \mathcal{E}^*) = 0$ *for $p \geq 1$ and every ℓ^1-geometric G-$C(X)$ module $\mathcal{E}$.*

The definition of ℓ^1-geometric G-$C(X)$ module is given in Section **??**. When X is a point our theorem reduces to Johnson's celebrated characterisation of amenability [6]. As a corollary we also obtain a cohomological characterisation of exactness for discrete groups, which answers a question of Higson, and which follows from our main result when X is the Stone-Čech compactification βG of the group G. In this case, $C(\beta G)$ can be identified with $\ell^\infty(G)$, and we obtain the following.

Corollary. *Let G be a countable discrete group. Then the following are equivalent.*

(1) *The group G is exact;*

(2) *The Johnson class $[J] \in H^1_b(G, N_0(G, \beta G)^{**})$ is trivial;*

(3) $H^p_b(G, \mathcal{E}^*) = 0$ *for $p \geq 1$ and every ℓ^1-geometric G-$\ell^\infty(G)$-module $\mathcal{E}$.*

This paper builds on the cohomological characterisation of property A developed in [3] and on the study of cohomological properties of exactness in [5].

2. Geometric Banach modules

Let $C(X)$ denote the space of real-valued continuous functions on X. For a function $f : G \to C(X)$ we shall denote by f_g the continuous function on X obtained by evaluating f at $g \in G$. We define the $\sup -\ell^1$ norm of f to be

$$\|f\|_{\infty,1} = \sup_{x \in X} \sum_{g \in G} |f_g(x)|,$$

and denote by V the Banach space of all functions on G with values in $C(X)$ that have finite norm. We introduce a Banach G-module associated to the action.

Definition 1. *Let $W_{00}(G, X)$ be the subspace of V consisting of all functions $f : G \to C(X)$ which have finite support and such that for some $c \in \mathbb{R}$, depending on f, $\sum_{g\in G} f_g = c1_X$, where 1_X denotes the constant function 1 on X. The closure of this space in the $\sup -\ell^1$-norm will be denoted $W_0(G, X)$.*

Let $\pi : W_{00}(G, X) \to \mathbb{R}$ be defined by $\sum_{g\in G} f_g = \pi(f)1_X$. The map π is continuous with respect to the $\sup -\ell^1$ norm and so extends to the closure $W_0(G, X)$; we denote its kernel by $N_0(G, X)$.

In the case of $X = \beta G$ and $C(\beta G) = \ell^\infty(G)$ the space $W_0(G, \beta G)$ was introduced in [5]. For every $g \in G$ we define the function $\delta_g \in W_{00}(G, X)$ by $\delta_g(h) = 1_X$ when $g = h$, and zero otherwise.

The G-action on X gives an isometric action of G on $C(X)$ in the usual way: for $g \in G$ and $f \in C(X)$, we have $(g \cdot f)(x) = f(g^{-1}x)$. The group G also acts isometrically on the space V in a natural way: for $g, h \in G, f \in V, x \in X$, we have $(gf)_h(x) = f_{g^{-1}h}(g^{-1}x) = (g \cdot f_{g^{-1}h})(x)$.

Since the summation map π is G-equivariant (we assume that the action of G on $\mathbb{R}$ is trivial) the action of G restricts to $W_{00}(G, X)$ and so by continuity it restricts to $W_0(G, X)$. We obtain a short exact sequence of G-vector spaces:

$$0 \to N_0(G, X) \to W_0(G, X) \xrightarrow{\pi} \mathbb{R} \to 0.$$

Definition 2. *Let $\mathcal{E}$ be a Banach space. We say that $\mathcal{E}$ is a $C(X)$-module if it is equipped with a contractive unital representation of the Banach algebra $C(X)$.*

If X is a G-space then a $C(X)$-module $\mathcal{E}$ is said to be a G-$C(X)$-module if the group G acts on $\mathcal{E}$ by isometries and the representation of $C(X)$ is G-equivariant.

Note that the fact that we will only ever consider unital representations of $C(X)$ means that there is no confusion between multiplying by a scalar or by the corresponding constant function. For instance, for $f \in W_0(G, X)$ multiplication by $\pi(f)$ agrees with multiplication by $\pi(f)1_X$.

Example 3. The space V is a G-$C(X)$-module. Indeed, for every $f \in V$ and $t \in C(X)$ we define $tf \in V$ by $(tf)_g(x) = t(x)f_g(x)$, for all $g \in G$. This action is well-defined as $\|tf\|_{\infty,1} \leq \|t\|_\infty \|f\|_{\infty,1}$; this also implies that the representation of $C(X)$ on V is contractive. As remarked above, the group G acts isometrically on V. The representation of $C(X)$ is clearly unital and also equivariant, since for every $g \in G$, $f \in V$ and $t \in C(X)$

$$(g(tf))_h(x) = (tf)_{g^{-1}h}(g^{-1}x) = t(g^{-1}x)f_{g^{-1}h}(g^{-1}x) = (g \cdot t)(x)(gf)_h(x)$$

Thus we have $g(tf) = (g \cdot t)(gf)$.

The equivariance of the summation map π implies that both $W_0(G, X)$ and $N_0(G, X)$ are G-invariant subspaces of V. Note however, that $W_0(G, X)$ is not invariant under the action of $C(X)$ defined above, as for $f \in W_0(G, X)$ and $t \in C(X)$ we have

$$\sum_{g \in G} (tf)_g(x) = \sum_{g \in G} t(x)f_g(x) = t(x) \sum_{g \in G} f_g(x) = ct(x).$$

However, the same calculation shows that the subspace $N_{00}(G, X)$ is invariant under the action of $C(X)$, and so is a G-$C(X)$-module, and hence so is its closure$N_0(G, X)$.

Let $\mathcal{E}$ be a G-$C(X)$-module, let $\mathcal{E}^*$ be the Banach dual of $\mathcal{E}$ and let $\langle -, - \rangle$ be the pairing between the two spaces. The induced actions of G and $C(X)$ on $\mathcal{E}^*$ are defined as follows. For $\alpha \in \mathcal{E}^*$, $g \in G$, $f \in C(X)$, and $v \in \mathcal{E}$ we let

$$\langle g\alpha, v \rangle = \langle \alpha, g^{-1}v \rangle, \qquad \langle f\alpha, v \rangle = \langle \alpha, fv \rangle.$$

Note that the action of $C(X)$ is well-defined since $C(X)$ is commutative. it is easy to check the following.

Lemma 4. *If $\mathcal{E}$ is a G-$C(X)$module, then so is $\mathcal{E}^*$.*

We will now introduce a geometric condition on Banach modules which will play the role of an orthogonality condition. To motivate the definition that follows, let us note that if f_1 and f_2 are functions with disjoint supports on a space X then (assuming that the relevant norms are finite) the sup-norm satisfies the identity $\|f_1 + f_2\|_\infty = \sup\{\|f_1\|_\infty, \|f_2\|_\infty\}$, while for the ℓ^1-norm we have $\|f_1 + f_2\|_{\ell^1} = \|f_1\|_{\ell^1} + \|f_2\|_{\ell^1}$.

Definition 5. *Let $\mathcal{E}$ be a Banach space and a $C(X)$-module. We say that v_1 and v_2 in $\mathcal{E}$ are disjointly supported if there exist $f_1, f_2 \in C(X)$ with disjoint supports such that $f_1v_1 = v_1$ and $f_2v_2 = v_2$.*

We say that the module $\mathcal{E}$ is ℓ^∞-geometric if, whenever v_1 and v_2 have disjoint supports, $\|v_1 + v_2\| = \sup\{\|v_1\|, \|v_2\|\}$.

We say that the module $\mathcal{E}$ is ℓ^1-geometric if for every two disjointly supported v_1 and v_2 in $\mathcal{E}$ $\|v_1 + v_2\| = \|v_1\| + \|v_2\|$.

If v_1 and v_2 are disjointly supported elements of $\mathcal{E}$ and f_1 and f_2 are as in the definition, then $f_1v_2 = f_1f_2v_2 = 0$, and similarly $f_2v_1 = 0$.

Note also that the functions f_1 and f_2 can be chosen to be of norm one in the supremum norm on $C(X)$. To see this, note that Tietze's extension theorem allows one to construct continuous functions f_1', f_2' on X which are of norm one, have disjoint supports and such that f_i' takes the value 1 on Supp f_i . Then $f_i'\phi_i = (f_i'f_i)\phi_i = f_i\phi_i = \phi_i$. Now replace f_i with f_i'.

Finally, if $f_1, f_2 \in C(X)$ have disjoint supports then, again by Tietze's extension theorem, $f_1\nu_1$ and $f_2\nu_2$ are disjointly supported for all $\nu_1, \nu_2 \in \mathcal{E}$.

Lemma 6. *If $\mathcal{E}$ is an ℓ^1-geometric module then $\mathcal{E}^*$ is ℓ^∞-geometric.*

If $\mathcal{E}$ is an ℓ^∞-geometric module then $\mathcal{E}^*$ is ℓ^1-geometric.

Proof. Let us assume that $\phi_1, \phi_2 \in \mathcal{E}^*$ are disjointly supported and let $f_1, f_2 \in C(X)$ be as in Definition 5, chosen to be of norm 1.

If $\mathcal{E}$ is ℓ^1-geometric, then for every vector $\nu \in \mathcal{E}$, $\|f_1\nu\| + \|f_2\nu\| = \|(f_1 + f_2)\nu\| \le \|\nu\|$. Furthermore,

$$\begin{aligned}
\|\phi_1 + \phi_2\| &= \sup_{\|\nu\|=1} |\langle \phi_1 + \phi_2, \nu\rangle| = \sup_{\|\nu\|=1} |\langle f_1\phi_1, \nu\rangle + \langle f_2\phi_2, \nu\rangle| \\
&= \sup_{\|\nu\|=1} |\langle \phi_1, f_1\nu\rangle + \langle \phi_2, f_2\nu\rangle| \\
&\le \sup_{\|\nu\|=1} (\|\phi_1\|\|f_1\nu\| + \|\phi_2\|\|f_2\nu\|) \\
&\le \sup\{\|\phi_1\|, \|\phi_2\|\} \sup_{\|\nu\|=1} (\|f_1\nu\| + \|f_2\nu\|) \\
&\le \sup\{\|\phi_1\|, \|\phi_2\|\}
\end{aligned}$$

Since $f_1\phi_2 = 0$ we have that

$$\|\phi_1\| = \|f_1(\phi_1 + \phi_2)\| \le \|f_1\|\|\phi_1 + \phi_2\| = \|\phi_1 + \phi_2\|.$$

Similarly, we have $\|\phi_2\| \le \|\phi_1 + \phi_2\|$, and the two estimates together ensure that $\|\phi_1 + \phi_2\| = \sup\{\|\phi_1\|, \|\phi_2\|\}$ as required.

For the second statement, let us assume that $\mathcal{E}$ is ℓ^∞-geometric and that $\phi_1, \phi_2 \in \mathcal{E}^*$ are disjointly supported. Then

$$\begin{aligned}
\|\phi_1\| + \|\phi_2\| &= \sup_{\|\nu_1\|, \|\nu_2\|=1} \langle \phi_1, \nu_1\rangle + \langle \phi_2, \nu_2\rangle \\
&= \sup_{\|\nu_1\|, \|\nu_2\|=1} \langle \phi_1, f_1\nu_1\rangle + \langle \phi_2, f_2\nu_2\rangle \\
&= \sup_{\|\nu_1\|, \|\nu_2\|=1} \langle \phi_1 + \phi_2, f_1\nu_1 + f_2\nu_2\rangle \\
&\le \sup_{\|\nu_1\|, \|\nu_2\|=1} \|\phi_1 + \phi_2\|\|f_1\nu_1 + f_2\nu_2\| \\
&\le \|\phi_1 + \phi_2\| \le \|\phi_1\| + \|\phi_2\|.
\end{aligned}$$

where the last inequality is just the triangle inequality, so the inequalities are equalities throughout and $\|\phi_1\| + \|\phi_2\| = \|\phi_1 + \phi_2\|$ as required. ∎

We have already established that $N_0(G,X)$ is a G-$C(X)$-module. Let ϕ^1 and ϕ^2 be disjointly supported elements of $N_0(G,X)$; this means that there exist disjointly supported functions f_1 and f_2 in $C(X)$ such that $\phi^i = f_i\phi^i$ for $i = 1, 2$. Then

$$\|\phi^1 + \phi^2\|_{\infty,1} = \|f_1\phi^1 + f_2\phi^2\| = \sup_{x\in X} \sum_{g\in G} |f_1(x)\phi^1_g(x) + f_2(x)\phi^2_g(x)|$$

We note that the two terms on the right are disjointly supported functions on X and so

$$\|\phi^1 + \phi^2\|_{\infty,1} = \sup_{x\in X} \left(\sum_{g\in G} |f_1(x)\phi^1_g(x)| + \sum_{g\in G} |f_2(x)\phi^2_g(x)| \right) = \sup(\|\phi^1\|_{\infty,1}, \|\phi^2\|_{\infty,1}).$$

Thus we obtain

Lemma 7. *The module $N_0(G,X)$ is ℓ^∞-geometric. Hence the dual $N_0(G,X)^*$ is ℓ^1-geometric and the double dual $N_0(G,X)^{**}$ is ℓ^∞-geometric.*

We now assume that $\mathcal{E}$ is an ℓ^1-geometric $C(X)$-module, so that its dual $\mathcal{E}^*$ is ℓ^∞-geometric.

Lemma 8. *Let $f_1, f_2 \in C(X)$ be non-negative functions such that $f_1 + f_2 \le 1_X$. Then for every $\phi_1, \phi_2 \in \mathcal{E}^*$*

$$\|f_1\phi_1 + f_2\phi_2\| \le \sup\{\|\phi_1\|, \|\phi_2\|\}.$$

Proof. Let $M \in \mathbb{N}$ and $\varepsilon = 1/M$. For $i = 1, 2$ define $f_{i,0} = \min\{f_i, \varepsilon\}$, $f_{i,1} = \min\{f_i - f_{i,0}, \varepsilon\}$, $f_{i,2} = \min\{f_i - f_{i,0} - f_{i,1}, \varepsilon\}$, and so on, to $f_{i,M-1}$.

Then $f_{i,j}(x) = 0$ iff $f_i(x) \le j\varepsilon$, so $f_{i,j} > 0$ iff $f_i(x) > j\varepsilon$ which implies that $\operatorname{Supp} f_{i,j} \subseteq f_i^{-1}([j\varepsilon, \infty))$. So for $j \ge 2$, $\operatorname{Supp}(f_{1,j}) \subseteq f_1^{-1}([j\epsilon, \infty))$ and $\operatorname{Supp} f_{2,M+1-j} \subseteq f_2^{-1}([(M + 1 - j)\epsilon, \infty))$.

If $x \in \operatorname{Supp}(f_{1,j}) \cap \operatorname{Supp}(f_{2,M+1-j})$ then $1 \ge f_1(x) + f_2(x) \ge j\varepsilon + (M + 1 - j)\varepsilon = 1 + \varepsilon$, so the two supports $\operatorname{Supp}(f_{1,j}), \operatorname{Supp}(f_{2,M+1-j})$ are disjoint.

We have that

$$f_1 = f_{1,0} + f_{1,1} + \sum_{j=2}^{M-1} f_{1,j}$$

$$f_2 = f_{2,0} + f_{2,1} + \sum_{j=2}^{M-1} f_{2,M+1-j}.$$

So using the fact that $\|f_{1,j}\phi_1 + f_{2,M+1-j}\phi_2\| \leq \sup\{\|f_{1,j}\phi_1\|, \|f_{2,M+1-j}\phi_2\|\} \leq \varepsilon \sup_i \|\phi_i\|$ we have the following estimate:

$$\begin{aligned}
\|f_1\phi_1 + f_2\phi_2\| &\leq \|(f_{1,0} + f_{1,1})\phi_1\| + \|(f_{2,0} + f_{2,1})\phi_2\| + \sum_{j=2}^{M} \|f_{1,j}\phi_1 + f_{2,M+1-j}\phi_2\| \\
&\leq 4\varepsilon \sup_j \|\phi_i\| + \sum_{j=2}^{M-1} \varepsilon \sup_i \|\phi_i\| \\
&= (4\varepsilon + (M-2)\varepsilon) \sup_i \|\phi_i\| \\
&= (1 + 2\varepsilon) \sup_i \|\phi_i\|.
\end{aligned}$$

□

Lemma 9. *Let* $f_1, \dots, f_N \in C(X)$, $f_i \geq 0$, $\sum_{i=1}^{N} f_i \leq 1_X$, $\phi_1, \dots, \phi_N \in \mathcal{E}^*$.

Then $\|\sum_i f_i\phi_i\| \leq \sup_{1,\dots,N} \|\phi_i\|$.

Proof. We proceed by induction. Assume that the statement is true for some N. Then let $f_0, f_1, \dots, f_N \in C(X)$, $f_i \geq 0$, $\sum_{i=1}^{N} f_i \leq 1_X$, and let $\phi_0, \phi_1, \dots, \phi_N \in \mathcal{E}^*$.

Let $f_1' = f_0 + f_1$ and leave the other functions unchanged. For $\delta > 0$ let

$$\phi_{1,\delta}' = \frac{1}{f_0 + f_1 + \delta}(f_0\phi_0 + f_1\phi_1).$$

Since we clearly have

$$\frac{f_0}{f_0 + f_1 + \delta} + \frac{f_1}{f_0 + f_1 + \delta} \leq 1_X$$

by the previous lemma we have that $\|\phi_{1,\delta}'\| \leq \sup\{\|\phi_0\|, \|\phi_1\|\}$, and so by induction

$$\|f_1'\phi_{1,\delta}' + f_2\phi_2 + \cdots + f_N\phi_N\| \leq \sup\{\|\phi_{1,\delta}'\|, \|\phi_2\|, \dots, \|\phi_N\|\} \leq \sup_{i=0,\dots,N} \|\phi_i\|.$$

Consider now

$$f_1'\phi_{1,\delta}' = \frac{(f_0 + f_1)}{f_0 + f_1 + \delta}(f_0\phi_0 + f_1\phi_1) = \frac{(f_0 + f_1)f_0}{f_0 + f_1 + \delta}\phi_0 + \frac{(f_0 + f_1)f_1}{f_0 + f_1 + \delta}\phi_1.$$

We note that for $i = 0, 1$

$$f_i - \frac{(f_0 + f_1)f_i}{f_0 + f_1 + \delta} = \frac{\delta f_i}{f_0 + f_1 + \delta} \leq \delta$$

and so $\frac{(f_0+f_1)f_i}{f_0+f_1+\delta}$ converges to f_i uniformly on X, as $\delta \to 0$, which implies that $f_1'\phi_{1,\delta}'$ converges to $f_0\phi_0 + f_1\phi_1$ in norm, and the lemma follows. □

Lemma 10. *If* $f_1, \ldots, f_N \in C(X)$ *(we do not assume that* $f_i \geq 0$*) are such that* $\sum_{i=1}^N |f_i| \leq 1_X$ *and* $\phi_1, \ldots, \phi_N \in \mathcal{E}^*$ *then*

$$\|\sum_{i=1}^N f_i \phi_i\| \leq 2 \sup_{i=1,\ldots,N} \|\phi_i\|.$$

Proof. If $f_i = f_i^+ - f_i^-$, then $|f_i| = f_i^+ + f_i^-$ and $\sum f_i^+ + \sum f_i^- \leq 1$.

Then by the previous lemma $\|\sum_{i=1}^N f_i^\pm \phi_i\| \leq \sup_{i=1,\ldots,N} \|\phi_i\|$ so

$$\|\sum f_i^+ \phi_i - \sum f_i^- \phi_i\| \leq 2 \sup_{i=1,\ldots,N} \|\phi_i\|.$$

□

3. Amenable actions and invariant means

In this section we will recall the definition of a topologically amenable action and characterise it in terms of the existence of a certain averaging operator. For our purposes the following definition, adapted from [4, Definition 4.3.1] is convenient.

Definition 11. *The action of* G *on* X *is amenable if and only if there exists a sequence of elements* $f^n \in W_{00}(G, X)$ *such that*

(1) $f_g^n \geq 0$ *in* $C(X)$ *for every* $n \in \mathbb{N}$ *and* $g \in G$*,*
(2) $\pi(f^n) = 1$ *for every* n*,*
(3) *for each* $g \in G$ *we have* $\|f^n - gf^n\|_V \to 0$*.*

Note that when X is a point the above conditions reduce to the definition of amenability of G. On the other hand, if $X = \beta G$, the Stone-Čech compactification of G then amenability of the natural action of G on X is equivalent to Yu's property A by a result of Higson and Roe [7].

Remark 12. In the above definition we may omit condition 1 at no cost, since given a sequence of functions satisfying conditions 2 and 3 we can make them positive by replacing each $f_g^n(x)$ by

$$\frac{|f_g^n(x)|}{\sum\limits_{h \in G} |f_h^n(x)|}.$$

Conditions 1 and 2 are now clear, while condition 3 follows from standard estimates (see e.g. [5, Lemma 4.9]).

The first definition of amenability of a group G given by von Neumann was in terms of the existence of an invariant mean on the group. The following definition gives a version of an invariant mean for an amenable action on a compact space.

Definition 13. *Let* G *be a countable group acting on a compact space* X *by homeomorphisms. A mean for the action is an element* $\mu \in W_0(G,X)^{**}$ *such that* $\mu(\pi) = 1$*. A mean* μ *is said to be invariant if* $\mu(g\varphi) = \mu(\varphi)$ *for every* $\varphi \in W_0(G,X)^*$.

We now state our first main result.

Theorem A. *Let* G *be a countable discrete group acting by homeomorphisms on a compact Hausdorff topological space* X*. The action is amenable if and only if there exists an invariant mean for the action.*

Proof. Let G act amenably on X and consider the sequence f^n provided by Definition 11. Each f^n satisfies $\|f^n\| = 1$. We now view the functions f^n as elements of the double dual $W_0(G,X)^{**}$. By the weak-* compactness of the unit ball there is a convergent subnet f^λ, and we define μ to be its weak-* limit. It is then easy to verify that μ is a mean. Since

$$|\langle f^\lambda - gf^\lambda, \varphi\rangle| \leq \|f^\lambda - gf^\lambda\|_V \|\varphi\|$$

and the right hand side tends to 0, we obtain $\mu(\varphi) = \mu(g\varphi)$.

Conversely, by Goldstine's theorem, (see, e.g., [8, Theorem 2.6.26]) as $\mu \in W_0(G,X)^{**}$, μ is the weak-* limit of a bounded net of elements $f^\lambda \in W_0(G,X)$. We note that we can choose f^λ in such a way that $\pi(f^\lambda) = 1$. Indeed, given f^λ with $\pi(f^\lambda) = c_\lambda \to \mu(\pi) = 1$ we replace each f^λ by

$$f^\lambda + (1 - c_\lambda)\delta_e.$$

Since $(1 - c_\lambda)\delta_e \to 0$ in norm in $W_0(G,X)$, μ is the weak-* limit of the net $f^\lambda + (1 - c_\lambda)\delta_e$ as required.

Since μ is invariant, we have that for every $g \in G$, $gf^\lambda \to g\mu = \mu$, so that $gf^\lambda - f^\lambda \to 0$ in the weak-* topology. However, for every $g \in G$, $gf^\lambda - f^\lambda \in W_0(G,X)$, and so the convergence is in fact in the weak topology on $W_0(G,X)$.

For every λ, we regard the family $(gf^\lambda - f^\lambda)_{g\in G}$ as an element of the product $\prod_{g\in G} W_0(G,X)$, noting that this sequence converges to 0 in the Tychonoff weak topology.

Now $\prod_{g\in G} W_0(G,X)$ is a Fréchet space in the Tychonoff norm topology, so by Mazur's theorem there exists a sequence f^n of convex combinations of f^λ such that $(gf^n - f^n)_{g\in G}$ converges to zero in the Fréchet topology. Thus there exists a sequence f^n of elements of $W_0(G,X)$ such that for every $g \in G$, $\|gf^n - f^n\| \to 0$ in $W_0(G,X)$.

The result then follows from Remark 12. □

4. Equivariant means on geometric modules

Given an invariant mean $\mu \in W_0(G,X)^{**}$ for the action of G on X and an ℓ^1-geometric G-$C(X)$ module $\mathcal{E}$, we define a G-equivariant averaging operator $\mu_{\mathcal{E}} : \ell^\infty(G, \mathcal{E}^*) \to \mathcal{E}^*$ which we will also refer to as an equivariant mean for the action.

To do so, following an idea from [3], we introduce a linear functional $\sigma_{\tau,\nu}$ on $W_{00}(G, X)$. Given a Banach space $\mathcal{E}$ define $\ell^\infty(G, \mathcal{E})$ to be the space of functions $f : G \to \mathcal{E}$ such that $\sup_{g\in G} \|f(g)\|_{\mathcal{E}} < \infty$. If G acts on $\mathcal{E}$ then the action of the group G on the space $\ell^\infty(G, \mathcal{E})$ is defined in an analogous way to the action of G on V, using the induced action of G on $\mathcal{E}$:

$$(g\tau)_h = g(\tau_{g^{-1}h}),$$

for $\tau \in \ell^\infty(G, \mathcal{E})$ and $g \in G$.

Let us assume that $\mathcal{E}$ is an ℓ^1-geometric G-$C(X)$ module, and let $\tau \in \ell^\infty(G, \mathcal{E}^*)$. Choose a vector $\nu \in \mathcal{E}$ and define a linear functional $\sigma_{\tau,\nu} : W_{00}(G, X) \to \mathbb{R}$ by

$$\sigma_{\tau,\nu}(f) = \langle \sum_{h\in G} f_h \tau_h, \nu\rangle \tag{1}$$

for every $f \in W_{00}(G, X)$. If we now use Lemma 10 together with the support condition required of elements of $W_{00}(G, X)$ then we have the estimate

$$|\sigma_{\tau,\nu}(f)| \leq \left\| \sum_h f_h \tau_h \right\| \|\nu\| \leq 2\|f\| \|\tau\| \|\nu\|.$$

This estimate completes the proof of the following.

Lemma 14. *Let $\mathcal{E}$ be an ℓ^1-geometric G-$C(X)$ module. For every $\tau \in \ell^\infty(G, \mathcal{E}^*)$ and every $\nu \in \mathcal{E}$ the linear functional $\sigma_{\tau,\nu}$ on $W_{00}(G, X)$ is continuous and so it extends to a continuous linear functional on $W_0(G, X)$.*

Lemma 15. *The map $\ell^\infty(G, \mathcal{E}^*) \times \mathcal{E} \to W_0(G, X)^*$ defined by $(\tau, \nu) \mapsto \sigma_{\tau,\nu}$ is G-equivariant.*

Proof.

$$\begin{aligned}
\sigma_{g\tau,g\nu}(f) &= \left\langle \sum_h f_h g(\tau_{g^{-1}h}), g\nu \right\rangle = \left\langle g \sum_h (g^{-1} \cdot f_h)\tau_{g^{-1}h}, g\nu \right\rangle \\
&= \left\langle \sum_h (g^{-1} \cdot f_h)\tau_{g^{-1}h}, \nu \right\rangle = \left\langle \sum_h (g^{-1} f)_{g^{-1}h}\tau_{g^{-1}h}, \nu \right\rangle \\
&= \sigma_{\tau,\nu}(g^{-1}f) = (g\sigma_{\tau,\nu})(f).
\end{aligned}$$

□

Definition 16. *Let $\mathcal{E}$ be an ℓ^1-geometric G-$C(X)$ module, and let $\mu \in W_0(G, X)^{**}$ be an invariant mean for the action. We define $\mu_{\mathcal{E}} : \ell^\infty(G, \mathcal{E}^*) \to \mathcal{E}^*$ by*

$$\langle \mu_{\mathcal{E}}(\tau), \nu\rangle = \langle \mu, \sigma_{\tau,\nu}\rangle,$$

for every $\tau \in \ell^\infty(G, \mathcal{E}^*)$, and $\nu \in \mathcal{E}$.

Lemma 17. *Let $\mathcal{E}$ be an ℓ^1-geometric G-$C(X)$ module, and let $\mu \in W_0(G, X)^{**}$ be an invariant mean for the action.*

(1) *The map $\mu_{\mathcal{E}}$ defined above is G-equivariant.*

(2) *If* $\tau \in \ell^\infty(G, \mathcal{E}^*)$ *is constant then* $\mu_{\mathcal{E}}(\tau) = \tau_e$.

Proof.

$$\begin{aligned}\langle \mu_{\mathcal{E}}(g\tau), \nu\rangle &= \mu(\sigma_{g\tau,\nu}) = \mu(g \cdot \sigma_{\tau,g^{-1}\nu}) = \mu(\sigma_{\tau,g^{-1}\nu}) \\ &= \langle \mu_{\mathcal{E}}(\tau), g^{-1}\nu\rangle = \langle g \cdot (\mu_{\mathcal{E}}(\tau)), \nu\rangle.\end{aligned}$$

If τ is constant then

$$\begin{aligned}\sigma_{\tau,\nu}(f) &= \left\langle \sum_h f_h\tau_h, \nu \right\rangle = \left\langle \left(\sum_h f_h\right) \tau_e, \nu \right\rangle \\ &= \langle (\pi(f)1_X)\tau_e, \nu\rangle = \langle \pi(f)\tau_e, \nu\rangle = \langle \tau_e, \nu\rangle\pi(f).\end{aligned}$$

So $\sigma_{\tau,\nu} = \langle \tau_e, \nu\rangle\pi$ and

$$\langle \mu_{\mathcal{E}}(\tau), \nu\rangle = \mu(\sigma_{\tau,\nu}) = \mu(\langle \tau_e, \nu\rangle\pi) = \langle \tau_e, \nu\rangle,$$

hence $\mu_{\mathcal{E}}(\tau) = \tau_e$. □

5. AMENABLE ACTIONS AND BOUNDED COHOMOLOGY

Let $\mathcal{E}$ be a Banach space equipped with an isometric action by G. Then we consider a cochain complex $C_b^m(G, \mathcal{E}^*)$ which in degree m consists of G-equivariant bounded cochains $\phi : G^{m+1} \to \mathcal{E}^*$ with values in the Banach dual $\mathcal{E}^*$ of $\mathcal{E}$ which is equipped with the natural differential d as in the homogeneous bar resolution. Bounded cohomology with coefficients in $\mathcal{E}^*$ will be denoted by $H_b^*(G, \mathcal{E}^*)$.

Definition 18. *Let* G *be a countable discrete group acting by homeomorphisms on a compact Hausdorff topological space* X*. The function*

$$J(g_0, g_1) = \delta_{g_1} - \delta_{g_0}$$

is a bounded cochain of degree 1 *with values in* $N_{00}(G, X)$*, and in fact it is a bounded cocycle and so represents a class in* $H_b^1(G, N_0(G, X)^{**}$*, where we regard* $N_{00}(G, X)$ *as a subspace of* $N_0(G, X)^{**}$*. We call* [J] *the Johnson class of the action.*

Theorem B. *Let* G *be a countable discrete group acting by homeomorphisms on a compact Hausdorff topological space* X*. Then the following are equivalent*

(1) *The action of* G *on* X *is topologically amenable.*

(2) *The class* $[J] \in H_b^1(G, N_0(G, X)^{**})$ *is trivial.*

(3) $H_b^p(G, \mathcal{E}^*) = 0$ *for* $p \geq 1$ *and every* ℓ^1*-geometric* G-$C(X)$ *module* $\mathcal{E}$.

Proof. We first show that (1) is equivalent to (2). The short exact sequence of G-modules

$$0 \to N_0(G, X) \to W_0(G, X) \xrightarrow{\pi} \mathbb{R} \to 0$$

leads, by taking double duals, to the short exact sequence

$$0 \to N_0(G,X)^{**} \to W_0(G,X)^{**} \to \mathbb{R} \to 0$$

which in turn gives rise to a long exact sequence in bounded cohomology

$$H^0_b(G, N_0(G,X)^{**}) \to H^0_b(G, W_0(G,X)^{**}) \to H^0_b(G,\mathbb{R}) \to H^1_b(G, N_0(G,X)^{**}) \to \dots$$

The Johnson class $[J]$ is the image of the class $[1] \in H^0_b(G,\mathbb{R})$ under the connecting homomorphism $d : H^0_b(G,\mathbb{R}) \to H^1(G, N_0(G,X)^{**})$, and so $[J] = 0$ if and only if $d[1] = 0$. By exactness of the cohomology sequence, this is equivalent to $[1] \in \operatorname{Im} \pi^{**}$, where $\pi^{**} : H^0_b(G, W_0(G,X)^{**}) \to H^0_b(G,\mathbb{R})$ is the map on cohomology induced by the summation map π. Since $H^0_b(G, W_0(G,X)^{**}) = (W_0(G,X)^{**})^G$ and $H^0_b(G,\mathbb{R}) = \mathbb{R}$ we have that $[J] = 0$ if and only if there exists an element $\mu \in W_0(G,X)^{**}$ such that $\mu = g\mu$ and $\mu(\pi) = 1$. Thus μ is an invariant mean for the action and the equivalence with amenability of the action follows from Theorem A.

We turn to the implication (1) implies (3). Since G acts amenably on X there is, by Theorem A, an invariant mean μ associated with the action. For every $h \in G$ and for every equivariant bounded cochain ϕ we define $s_h\phi : G^p \to \mathcal{E}^*$ by $s_h\phi(g_0, \dots, g_{p-1}) = \phi(g, g_0, \dots, g_{p-1})$; we note that for fixed h, $s_h\phi$ is not equivariant in general. However, the map s_h does satisfy the identity $ds_h + s_h d = 1$ for every $h \in G$, and we will now construct an equivariant contracting homotopy, adapting an averaging procedure introduced in [3].

For $\phi \in C^p_b(G,\mathcal{E}^*)$ let $\widehat{\phi} : G^p \to \ell^\infty(G,\mathcal{E}^*)$ be defined by $\widehat{\phi}(\mathbf{g})(h) = s_h\phi(\mathbf{g})$, for $\mathbf{g} = (g_0, \dots g_{p-1})$.

Note that for every $k, h \in G$,

$$\begin{aligned}
\widehat{\phi}(kg_0, \dots, kg_{p-1})(h) &= \phi(h, kg_0, \dots, kg_{p-1}) = k(\phi(k^{-1}h, g_0, \dots, g_{p-1}))\\
&= k(\widehat{\phi}(g_0, \dots, g_{p-1})(k^{-1}h))\\
&= (k(\widehat{\phi}(g_0, \dots, g_{p-1})))(h)
\end{aligned}$$

so $\widehat{\phi}(k\mathbf{g}) = k(\widehat{\phi}(\mathbf{g}))$.

We can now define a map $s : C^p(G,\mathcal{E}^*) \to C^{p-1}(G,\mathcal{E}^*)$:

$$s\phi(\mathbf{g}) = \mu_{\mathcal{E}}(\widehat{\phi}(\mathbf{g})),$$

where $\mu_{\mathcal{E}} : \ell^\infty(G,\mathcal{E}^*) \to \mathcal{E}^*$ is the map defined in Lemma 17 using the invariant mean μ. Note that $\|\mu_{\mathcal{E}}\| \le 2\|\mu\|$, and $\|\widehat{\phi}(\mathbf{g})\| \le \sup\{\|\phi(\mathbf{k})\| \mid \mathbf{k} \in G^{p+1}\}$. Hence $s\phi$ is bounded.

For every cochain ϕ, $k(s\phi) = s(k\phi) = s\phi$ since $\widehat{\phi}$ and $\mu_{\mathcal{E}}$ are equivariant.

The map s provides a contracting homotopy for the complex $C^*_b(G,\mathcal{E}^*)$ which can be seen as follows. As $\mu_{\mathcal{E}} : \ell^\infty(G,\mathcal{E}^*) \to \mathcal{E}^*$ is a linear operator it follows that for a given $\phi \in C^p_b(G,\mathcal{E}^*)$, and a $p+1$-tuple of arguments $\mathbf{k} = (k_0, \dots, k_p)$, $ds\phi$ is obtained by applying the mean $\mu_{\mathcal{E}}$ to the map $g \mapsto ds_g\phi(\mathbf{k})$, while $sd\phi$ is obtained by applying $\mu_{\mathcal{E}}$ to the function $g \mapsto s_g d\phi(\mathbf{k})$. Thus

$$(sd + ds)\phi(\mathbf{k}) = \mu_{\mathcal{E}}(g \mapsto (ds_g + s_g d)\phi(\mathbf{k})).$$

Given that $ds_g + s_g d = 1$ for every $g \in G$, for every $\mathbf{g} \in G^{p+1}$ the function $g \mapsto (ds_g + s_g d)\phi(\mathbf{k}) = \phi(\mathbf{k}) \in \mathcal{E}^*$ is constant, and so by Lemma 17,

$$(sd + ds)\phi(\mathbf{k}) = (ds_e + s_e d)\phi(\mathbf{k}) = \phi(\mathbf{k}).$$

Thus $sd + ds = 1$, as required.

Collecting these results together, we have proved that (1) implies (3).

The fact that (3) implies (2), follows from the fact that $N_0(G, X)^*$ is an ℓ^1-geometric G-$C(X)$-module, proved in Lemma 7.

□

REFERENCES

[1] C. Anantharaman-Delaroche, J. Renault, Amenable groupoids, Monographies de L'Enseignement Mathématique, 36. L'Enseignement Mathématique, Geneva, 2000.

[2] J. Block and S. Weinberger, Aperiodic tilings, positive scalar curvature and amenability of spaces. J. Amer. Math. Soc. 5 (1992), no. 4, 907–918.

[3] J. Brodzki, G. A. Niblo and N. J. Wright, A cohomological characterisation of Yu's Property A for metric spaces. Preprint, University of Southampton ePrints November 2009: arXiv:1002.5040.

[4] N. P. Brown, N. Ozawa, C^*-algebras, and finite dimensional approximations, Graduate Studies in Mathematics, 88, AMS, Rhode Island 2008.

[5] R. G. Douglas and P. W. Nowak, Invariant expectations and vanishing of bounded cohomology for exact groups. Preprint, arXiv:1001.0718v3.

[6] B. E. Johnson, Cohomology of Banach Algebras, Memoirs of the AMS Number 127, 1972, AMS, Providence, Rhode Island.

[7] N. Higson, J. Roe, Amenable group actions and the Novikov conjecture, J. Reine Angew. Math. 519 (2000), 143–153.

[8] R. E. Megginson, An Introduction to Banach Space Theory, (Graduate Texts in Mathematics 183), Springer Verlag New York, 1998.

[9] N. Ozawa, Amenable actions and exactness for discrete groups, C.R. Acad. Sci. Paris Ser. I Math 330 (2000), 691-695.

[10] G. Yu, The Coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space, Inventiones Math. **139** (2000), 201–240.

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AMENABLE ACTIONS, INVARIANT MEANS AND BOUNDED COHOMOLOGY

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ABSTRACT. We show that topological amenability of an action of a countable discrete group on a compact space is equivalent to the existence of an invariant mean for the action. We prove also that this is equivalent to vanishing of bounded cohomology for a class of Banach G-modules associated to the action, as well as to vanishing of a specific cohomology class. In the case when the compact space is a point our result reduces to a classic theorem of B.E. Johnson characterising amenability of groups. In the case when the compact space is the Stone-Čech compactification of the group we obtain a cohomological characterisation of exactness for the group, answering a question of Higson.

1. INTRODUCTION

An invariant mean on a countable discrete group G is a positive linear functional on $\ell^\infty(G)$ which is normalised by the requirement that it pairs with the constant function 1 to give 1, and which is fixed by the natural action of G on the space $\ell^\infty(G)^*$. A group is said to be amenable if it admits an invariant mean. The notion of an amenable action of a group on a topological space, studied by Anantharaman-Delaroche and Renault [1], generalises the concept of amenability, and arises naturally in many areas of mathematics. For example, a group acts amenably on a point if and only if it is amenable, while every hyperbolic group acts amenably on its Gromov boundary.

In this paper we introduce the notion of an invariant mean for a topological action and prove that the existence of such a mean characterises amenability of the action. Moreover, we use the existence of the mean to prove vanishing of bounded cohomology of G with coefficients in a suitable class of Banach G modules, and conversely we prove that vanishing of these cohomology groups characterises amenability of the action. This generalises the results of Johnson [6] on bounded cohomology for amenable groups.

Another generalisation of amenability, this time for metric spaces, was given by Yu [10] with the definition of property A. Higson and Roe [7] proved a remarkable result that unifies the two approaches: A finitely generated discrete group G (regarded as a metric space) has Yu's property A if and only if the action of G on its Stone-Čech compactification βG is topologically amenable, and this is true if and only if G acts amenably on any compact space. Ozawa proved [9] that such groups are exact, and indeed property A and exactness are equivalent for countable discrete groups equipped with a proper left-invariant metric.

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To generalise the concept of invariant mean to the context of a topological action, we introduce a Banach G-module $W_0(G, X)$ which is an analogue of $\ell^1(G)$, encoding both the group and the space on which it acts. Taking the dual and double dual of this space we obtain analogues of $\ell^\infty(G)$ and $\ell^\infty(G)^*$. A mean for the action is an element $\mu \in W_0(G, X)^{**}$ satisfying the normalisation condition $\mu(\pi) = 1$, where the element π is a summation operator, corresponding to the pairing of $\ell^1(G)$ with the constant function 1 in $\ell^\infty(G)$. A mean μ is said to be invariant if $\mu(g \cdot \varphi) = \mu(\varphi)$ for every $\varphi \in W_0(G, X)^*$, (Definition 13).

With these notions in place we give the following very natural characterisation of amenable actions.

Theorem A. *Let* G *be a countable discrete group acting by homeomorphisms on a compact Hausdorff topological space* X*. The action is amenable if and only if there exists an invariant mean for the action.*

We then turn to the question of a cohomological characterisation of amenable actions. Given an action of a countable discrete group G on a compact space X by homeomorphisms we introduce a submodule $N_0(G, X)$ of $W_0(G, X)$ associated to the action and which is analogous to the submodule $\ell^1_0(G)$ of $\ell^1(G)$ consisting of all functions of sum 0. Indeed when X is a point these modules coincide. We also define a cohomology class $[J]$, called the Johnson class of the action, which lives in the first bounded cohomology group of G with coefficients in the module $N_0(G, X)^{**}$. We have the following theorem.

Theorem B. *Let* G *be a countable discrete group acting by homeomorphisms on a compact Hausdorff topological space* X*. Then the following are equivalent*

(1) *The action of* G *on* X *is topologically amenable.*

(2) *The class* $[J] \in H^1_b(G, N_0(G, X)^{**})$ *is trivial.*

(3) $H^p_b(G, \mathcal{E}^*) = 0$ *for* $p \geq 1$ *and every* ℓ^1*-geometric* G-$C(X)$ *module* $\mathcal{E}$*.*

The definition of ℓ^1-geometric G-$C(X)$ module is given in Section **??**. When X is a point our theorem reduces to Johnson's celebrated characterisation of amenability [6]. As a corollary we also obtain a cohomological characterisation of exactness for discrete groups, which answers a question of Higson, and which follows from our main result when X is the Stone-Čech compactification βG of the group G. In this case, $C(\beta G)$ can be identified with $\ell^\infty(G)$, and we obtain the following.

Corollary. *Let* G *be a countable discrete group. Then the following are equivalent.*

(1) *The group* G *is exact;*

(2) *The Johnson class* $[J] \in H^1_b(G, N_0(G, \beta G)^{**})$ *is trivial;*

(3) $H^p_b(G, \mathcal{E}^*) = 0$ *for* $p \geq 1$ *and every* ℓ^1*-geometric* G-$\ell^\infty(G)$*-module* $\mathcal{E}$*.*

This paper builds on the cohomological characterisation of property A developed in [3] and on the study of cohomological properties of exactness in [5].

2. Geometric Banach modules

Let $C(X)$ denote the space of real-valued continuous functions on X. For a function $f : G \to C(X)$ we shall denote by f_g the continuous function on X obtained by evaluating f at $g \in G$. We define the sup $-\ell^1$ norm of f to be

$$\|f\|_{\infty,1} = \sup_{x \in X} \sum_{g \in G} |f_g(x)|,$$

and denote by V the Banach space of all functions on G with values in $C(X)$ that have finite norm. We introduce a Banach G-module associated to the action.

Definition 1. *Let $W_{00}(G, X)$ be the subspace of V consisting of all functions $f : G \to C(X)$ which have finite support and such that for some $c \in \mathbb{R}$, depending on f, $\sum_{g \in G} f_g = c 1_X$, where 1_X denotes the constant function 1 on X. The closure of this space in the sup $-\ell^1$-norm will be denoted $W_0(G, X)$.*

Let $\pi : W_{00}(G, X) \to \mathbb{R}$ be defined by $\sum_{g \in G} f_g = \pi(f) 1_X$. The map π is continuous with respect to the sup $-\ell^1$ norm and so extends to the closure $W_0(G, X)$; we denote its kernel by $N_0(G, X)$.

In the case of $X = \beta G$ and $C(\beta G) = \ell^\infty(G)$ the space $W_0(G, \beta G)$ was introduced in [5]. For every $g \in G$ we define the function $\delta_g \in W_{00}(G, X)$ by $\delta_g(h) = 1_X$ when $g = h$, and zero otherwise.

The G-action on X gives an isometric action of G on $C(X)$ in the usual way: for $g \in G$ and $f \in C(X)$, we have $(g \cdot f)(x) = f(g^{-1}x)$. The group G also acts isometrically on the space V in a natural way: for $g, h \in G, f \in V, x \in X$, we have $(gf)_h(x) = f_{g^{-1}h}(g^{-1}x) = (g \cdot f_{g^{-1}h})(x)$.

Since the summation map π is G-equivariant (we assume that the action of G on $\mathbb{R}$ is trivial) the action of G restricts to $W_{00}(G, X)$ and so by continuity it restricts to $W_0(G, X)$. We obtain a short exact sequence of G-vector spaces:

$$0 \to N_0(G, X) \to W_0(G, X) \xrightarrow{\pi} \mathbb{R} \to 0.$$

Definition 2. *Let $\mathcal{E}$ be a Banach space. We say that $\mathcal{E}$ is a $C(X)$-module if it is equipped with a contractive unital representation of the Banach algebra $C(X)$.*

If X is a G-space then a $C(X)$-module $\mathcal{E}$ is said to be a G-$C(X)$-module if the group G acts on $\mathcal{E}$ by isometries and the representation of $C(X)$ is G-equivariant.

Note that the fact that we will only ever consider unital representations of $C(X)$ means that there is no confusion between multiplying by a scalar or by the corresponding constant function. For instance, for $f \in W_0(G, X)$ multiplication by $\pi(f)$ agrees with multiplication by $\pi(f)1_X$.

Example 3. The space V is a G-$C(X)$-module. Indeed, for every $f \in V$ and $t \in C(X)$ we define $tf \in V$ by $(tf)_g(x) = t(x)f_g(x)$, for all $g \in G$. This action is well-defined as $\|tf\|_{\infty,1} \leq \|t\|_\infty \|f\|_{\infty,1}$; this also implies that the representation of $C(X)$ on V is contractive. As remarked above, the group G acts isometrically on V. The representation of $C(X)$ is clearly unital and also equivariant, since for every $g \in G$, $f \in V$ and $t \in C(X)$

$$(g(tf))_h(x) = (tf)_{g^{-1}h}(g^{-1}x) = t(g^{-1}x)f_{g^{-1}h}(g^{-1}x) = (g \cdot t)(x)(gf)_h(x)$$

Thus we have $g(tf) = (g \cdot t)(gf)$.

The equivariance of the summation map π implies that both $W_0(G, X)$ and $N_0(G, X)$ are G-invariant subspaces of V. Note however, that $W_0(G, X)$ is not invariant under the action of $C(X)$ defined above, as for $f \in W_0(G, X)$ and $t \in C(X)$ we have

$$\sum_{g \in G} (tf)_g(x) = \sum_{g \in G} t(x)f_g(x) = t(x) \sum_{g \in G} f_g(x) = ct(x).$$

However, the same calculation shows that the subspace $N_{00}(G, X)$ is invariant under the action of $C(X)$, and so is a G-$C(X)$-module, and hence so is its closure$N_0(G, X)$.

Let $\mathcal{E}$ be a G-$C(X)$-module, let $\mathcal{E}^*$ be the Banach dual of $\mathcal{E}$ and let $\langle -, - \rangle$ be the pairing between the two spaces. The induced actions of G and $C(X)$ on $\mathcal{E}^*$ are defined as follows. For $\alpha \in \mathcal{E}^*$, $g \in G$, $f \in C(X)$, and $v \in \mathcal{E}$ we let

$$\langle g\alpha, v \rangle = \langle \alpha, g^{-1}v \rangle, \qquad \langle f\alpha, v \rangle = \langle \alpha, fv \rangle.$$

Note that the action of $C(X)$ is well-defined since $C(X)$ is commutative. it is easy to check the following.

Lemma 4. *If $\mathcal{E}$ is a G-$C(X)$module, then so is $\mathcal{E}^*$.*

We will now introduce a geometric condition on Banach modules which will play the role of an orthogonality condition. To motivate the definition that follows, let us note that if f_1 and f_2 are functions with disjoint supports on a space X then (assuming that the relevant norms are finite) the sup-norm satisfies the identity $\|f_1 + f_2\|_\infty = \sup\{\|f_1\|_\infty, \|f_2\|_\infty\}$, while for the ℓ^1-norm we have $\|f_1 + f_2\|_{\ell^1} = \|f_1\|_{\ell^1} + \|f_2\|_{\ell^1}$.

Definition 5. *Let $\mathcal{E}$ be a Banach space and a $C(X)$-module. We say that v_1 and v_2 in $\mathcal{E}$ are disjointly supported if there exist $f_1, f_2 \in C(X)$ with disjoint supports such that $f_1v_1 = v_1$ and $f_2v_2 = v_2$.*

We say that the module $\mathcal{E}$ is ℓ^∞-geometric if, whenever v_1 and v_2 have disjoint supports, $\|v_1 + v_2\| = \sup\{\|v_1\|, \|v_2\|\}$.

We say that the module $\mathcal{E}$ is ℓ^1-geometric if for every two disjointly supported v_1 and v_2 in $\mathcal{E}$ $\|v_1 + v_2\| = \|v_1\| + \|v_2\|$.

If v_1 and v_2 are disjointly supported elements of $\mathcal{E}$ and f_1 and f_2 are as in the definition, then $f_1v_2 = f_1f_2v_2 = 0$, and similarly $f_2v_1 = 0$.

Note also that the functions f_1 and f_2 can be chosen to be of norm one in the supremum norm on $C(X)$. To see this, note that Tietze's extension theorem allows one to construct continuous functions f_1', f_2' on X which are of norm one, have disjoint supports and such that f_i' takes the value 1 on $\mathrm{Supp}\, f_i$. Then $f_i'\phi_i = (f_i' f_i)\phi_i = f_i\phi_i = \phi_i$. Now replace f_i with f_i'.

Finally, if $f_1, f_2 \in C(X)$ have disjoint supports then, again by Tietze's extension theorem, $f_1\nu_1$ and $f_2\nu_2$ are disjointly supported for all $\nu_1, \nu_2 \in \mathcal{E}$.

Lemma 6. *If $\mathcal{E}$ is an ℓ^1-geometric module then $\mathcal{E}^*$ is ℓ^∞-geometric.*

If $\mathcal{E}$ is an ℓ^∞-geometric module then $\mathcal{E}^*$ is ℓ^1-geometric.

Proof. Let us assume that $\phi_1, \phi_2 \in \mathcal{E}^*$ are disjointly supported and let $f_1, f_2 \in C(X)$ be as in Definition 5, chosen to be of norm 1.

If $\mathcal{E}$ is ℓ^1-geometric, then for every vector $\nu \in \mathcal{E}$, $\|f_1\nu\| + \|f_2\nu\| = \|(f_1+f_2)\nu\| \le \|\nu\|$. Furthermore,

$$\begin{aligned}
\|\phi_1+\phi_2\| &= \sup_{\|\nu\|=1} |\langle \phi_1+\phi_2, \nu\rangle| = \sup_{\|\nu\|=1} |\langle f_1\phi_1, \nu\rangle + \langle f_2\phi_2, \nu\rangle| \\
&= \sup_{\|\nu\|=1} |\langle \phi_1, f_1\nu\rangle + \langle \phi_2, f_2\nu\rangle| \\
&\le \sup_{\|\nu\|=1} \left(\|\phi_1\|\|f_1\nu\| + \|\phi_2\|\|f_2\nu\|\right) \\
&\le \sup\{\|\phi_1\|, \|\phi_2\|\} \sup_{\|\nu\|=1} \left(\|f_1\nu\| + \|f_2\nu\|\right) \\
&\le \sup\{\|\phi_1\|, \|\phi_2\|\}
\end{aligned}$$

Since $f_1\phi_2 = 0$ we have that

$$\|\phi_1\| = \|f_1(\phi_1+\phi_2)\| \le \|f_1\|\|\phi_1+\phi_2\| = \|\phi_1+\phi_2\|.$$

Similarly, we have $\|\phi_2\| \le \|\phi_1+\phi_2\|$, and the two estimates together ensure that $\|\phi_1+\phi_2\| = \sup\{\|\phi_1\|, \|\phi_2\|\}$ as required.

For the second statement, let us assume that $\mathcal{E}$ is ℓ^∞-geometric and that $\phi_1, \phi_2 \in \mathcal{E}^*$ are disjointly supported. Then

$$\begin{aligned}
\|\phi_1\| + \|\phi_2\| &= \sup_{\|\nu_1\|,\|\nu_2\|=1} \langle \phi_1, \nu_1\rangle + \langle \phi_2, \nu_2\rangle \\
&= \sup_{\|\nu_1\|,\|\nu_2\|=1} \langle \phi_1, f_1\nu_1\rangle + \langle \phi_2, f_2\nu_2\rangle \\
&= \sup_{\|\nu_1\|,\|\nu_2\|=1} \langle \phi_1+\phi_2, f_1\nu_1 + f_2\nu_2\rangle \\
&\le \sup_{\|\nu_1\|,\|\nu_2\|=1} \|\phi_1+\phi_2\|\|f_1\nu_1+f_2\nu_2\| \\
&\le \|\phi_1+\phi_2\| \le \|\phi_1\| + \|\phi_2\|.
\end{aligned}$$

where the last inequality is just the triangle inequality, so the inequalities are equalities throughout and $\|\phi_1\| + \|\phi_2\| = \|\phi_1+\phi_2\|$ as required. □

We have already established that $N_0(G, X)$ is a G-$C(X)$-module. Let ϕ^1 and ϕ^2 be disjointly supported elements of $N_0(G, X)$; this means that there exist disjointly supported functions f_1 and f_2 in $C(X)$ such that $\phi^i = f_i\phi^i$ for $i = 1, 2$. Then

$$\|\phi^1 + \phi^2\|_{\infty,1} = \|f_1\phi^1 + f_2\phi^2\| = \sup_{x\in X}\sum_{g\in G}|f_1(x)\phi^1_g(x) + f_2(x)\phi^2_g(x)|$$

We note that the two terms on the right are disjointly supported functions on X and so

$$\|\phi^1 + \phi^2\|_{\infty,1} = \sup_{x\in X}\left(\sum_{g\in G}|f_1(x)\phi^1_g(x)| + \sum_{g\in G}|f_2(x)\phi^2_g(x)|\right) = \sup(\|\phi^1\|_{\infty,1}, \|\phi^2\|_{\infty,1}).$$

Thus we obtain

Lemma 7. *The module $N_0(G, X)$ is ℓ^∞-geometric. Hence the dual $N_0(G, X)^*$ is ℓ^1-geometric and the double dual $N_0(G, X)^{**}$ is ℓ^∞-geometric.*

We now assume that $\mathcal{E}$ is an ℓ^1-geometric $C(X)$-module, so that its dual $\mathcal{E}^*$ is ℓ^∞-geometric.

Lemma 8. *Let $f_1, f_2 \in C(X)$ be non-negative functions such that $f_1 + f_2 \leq 1_X$. Then for every $\phi_1, \phi_2 \in \mathcal{E}^*$*

$$\|f_1\phi_1 + f_2\phi_2\| \leq \sup\{\|\phi_1\|, \|\phi_2\|\}.$$

Proof. Let $M \in \mathbb{N}$ and $\varepsilon = 1/M$. For $i = 1, 2$ define $f_{i,0} = \min\{f_i, \varepsilon\}$, $f_{i,1} = \min\{f_i - f_{i,0}, \varepsilon\}$, $f_{i,2} = \min\{f_i - f_{i,0} - f_{i,1}, \varepsilon\}$, and so on, to $f_{i,M-1}$.

Then $f_{i,j}(x) = 0$ iff $f_i(x) \leq j\varepsilon$, so $f_{i,j} > 0$ iff $f_i(x) > j\varepsilon$ which implies that $\mathrm{Supp}\, f_{i,j} \subseteq f_i^{-1}([j\varepsilon, \infty))$. So for $j \geq 2$, $\mathrm{Supp}(f_{1,j}) \subseteq f_1^{-1}([j\epsilon, \infty))$ and $\mathrm{Supp}\, f_{2,M+1-j} \subseteq f_2^{-1}([(M + 1 - j)\epsilon, \infty))$.

If $x \in \mathrm{Supp}(f_{1,j}) \cap \mathrm{Supp}(f_{2,M+1-j})$ then $1 \geq f_1(x) + f_2(x) \geq j\varepsilon + (M + 1 - j)\varepsilon = 1 + \varepsilon$, so the two supports $\mathrm{Supp}(f_{1,j}), \mathrm{Supp}(f_{2,M+1-j})$ are disjoint.

We have that

$$f_1 = f_{1,0} + f_{1,1} + \sum_{j=2}^{M-1} f_{1,j}$$
$$f_2 = f_{2,0} + f_{2,1} + \sum_{j=2}^{M-1} f_{2,M+1-j}.$$

So using the fact that $\|f_{1,j}\phi_1 + f_{2,M+1-j}\phi_2\| \leq \sup\{\|f_{1,j}\phi_1\|, \|f_{2,M+1-j}\phi_2\|\} \leq \varepsilon \sup_i \|\phi_i\|$ we have the following estimate:

$$\begin{aligned}
\|f_1\phi_1 + f_2\phi_2\| &\leq \|(f_{1,0} + f_{1,1})\phi_1\| + \|(f_{2,0} + f_{2,1})\phi_2\| + \sum_{j=2}^{M} \|f_{1,j}\phi_1 + f_{2,M+1-j}\phi_2\| \\
&\leq 4\varepsilon \sup_j \|\phi_i\| + \sum_{j=2}^{M-1} \varepsilon \sup_i \|\phi_i\| \\
&= (4\varepsilon + (M-2)\varepsilon) \sup_i \|\phi_i\| \\
&= (1 + 2\varepsilon) \sup_i \|\phi_i\|.
\end{aligned}$$

□

Lemma 9. *Let* $f_1, \ldots, f_N \in C(X)$, $f_i \geq 0$, $\sum_{i=1}^N f_i \leq 1_X$, $\phi_1, \ldots, \phi_N \in \mathcal{E}^*$.

Then $\|\sum_i f_i\phi_i\| \leq \sup_{1,\ldots,N} \|\phi_i\|$.

Proof. We proceed by induction. Assume that the statement is true for some N. Then let $f_0, f_1, \ldots, f_N \in C(X)$, $f_i \geq 0$, $\sum_{i=1}^N f_i \leq 1_X$, and let $\phi_0, \phi_1, \ldots, \phi_N \in \mathcal{E}^*$.

Let $f_1' = f_0 + f_1$ and leave the other functions unchanged. For $\delta > 0$ let

$$\phi_{1,\delta}' = \frac{1}{f_0 + f_1 + \delta}(f_0\phi_0 + f_1\phi_1).$$

Since we clearly have

$$\frac{f_0}{f_0 + f_1 + \delta} + \frac{f_1}{f_0 + f_1 + \delta} \leq 1_X$$

by the previous lemma we have that $\|\phi_{1,\delta}'\| \leq \sup\{\|\phi_0\|, \|\phi_1\|\}$, and so by induction

$$\|f_1'\phi_{1,\delta}' + f_2\phi_2 + \cdots + f_N\phi_N\| \leq \sup\{\|\phi_{1,\delta}'\|, \|\phi_2\|, \ldots, \|\phi_N\|\} \leq \sup_{i=0,\ldots,N} \|\phi_i\|.$$

Consider now

$$f_1'\phi_{1,\delta}' = \frac{(f_0 + f_1)}{f_0 + f_1 + \delta}(f_0\phi_0 + f_1\phi_1) = \frac{(f_0 + f_1)f_0}{f_0 + f_1 + \delta}\phi_0 + \frac{(f_0 + f_1)f_1}{f_0 + f_1 + \delta}\phi_1.$$

We note that for $i = 0, 1$

$$f_i - \frac{(f_0 + f_1)f_i}{f_0 + f_1 + \delta} = \frac{\delta f_i}{f_0 + f_1 + \delta} \leq \delta$$

and so $\frac{(f_0+f_1)f_i}{f_0+f_1+\delta}$ converges to f_i uniformly on X, as $\delta \to 0$, which implies that $f_1'\phi_{1,\delta}'$ converges to $f_0\phi_0 + f_1\phi_1$ in norm, and the lemma follows. □

Lemma 10. *If* $f_1, \ldots, f_N \in C(X)$ *(we do not assume that* $f_i \geq 0$*) are such that* $\sum_{i=1}^N |f_i| \leq 1_X$ *and* $\phi_1, \ldots, \phi_N \in \mathcal{E}^*$ *then*

$$\|\sum_{i=1}^N f_i \phi_i\| \leq 2 \sup_{i=1,\ldots,N} \|\phi_i\|.$$

Proof. If $f_i = f_i^+ - f_i^-$, then $|f_i| = f_i^+ + f_i^-$ and $\sum f_i^+ + \sum f_i^- \leq 1$.

Then by the previous lemma $\|\sum_{i=1}^N f_i^{\pm} \phi_i\| \leq \sup_{i=1,\ldots,N} \|\phi_i\|$ so

$$\|\sum f_i^+ \phi_i - \sum f_i^- \phi_i\| \leq 2 \sup_{i=1,\ldots,N} \|\phi_i\|.$$

□

3. Amenable actions and invariant means

In this section we will recall the definition of a topologically amenable action and characterise it in terms of the existence of a certain averaging operator. For our purposes the following definition, adapted from [4, Definition 4.3.1] is convenient.

Definition 11. *The action of* G *on* X *is amenable if and only if there exists a sequence of elements* $f^n \in W_{00}(G, X)$ *such that*

(1) $f_g^n \geq 0$ *in* $C(X)$ *for every* $n \in \mathbb{N}$ *and* $g \in G$,
(2) $\pi(f^n) = 1$ *for every* n,
(3) *for each* $g \in G$ *we have* $\|f^n - gf^n\|_V \to 0$.

Note that when X is a point the above conditions reduce to the definition of amenability of G. On the other hand, if $X = \beta G$, the Stone-Čech compactification of G then amenability of the natural action of G on X is equivalent to Yu's property A by a result of Higson and Roe [7].

Remark 12. In the above definition we may omit condition 1 at no cost, since given a sequence of functions satisfying conditions 2 and 3 we can make them positive by replacing each $f_g^n(x)$ by

$$\frac{|f_g^n(x)|}{\sum_{h \in G} |f_h^n(x)|}.$$

Conditions 1 and 2 are now clear, while condition 3 follows from standard estimates (see e.g. [5, Lemma 4.9]).

The first definition of amenability of a group G given by von Neumann was in terms of the existence of an invariant mean on the group. The following definition gives a version of an invariant mean for an amenable action on a compact space.

Definition 13. *Let* G *be a countable group acting on a compact space* X *by homeomorphisms. A mean for the action is an element* $\mu \in W_0(G, X)^{**}$ *such that* $\mu(\pi) = 1$*. A mean* μ *is said to be invariant if* $\mu(g\varphi) = \mu(\varphi)$ *for every* $\varphi \in W_0(G, X)^*$.

We now state our first main result.

Theorem A. *Let* G *be a countable discrete group acting by homeomorphisms on a compact Hausdorff topological space* X*. The action is amenable if and only if there exists an invariant mean for the action.*

Proof. Let G act amenably on X and consider the sequence f^n provided by Definition 11. Each f^n satisfies $\|f^n\| = 1$. We now view the functions f^n as elements of the double dual $W_0(G, X)^{**}$. By the weak-* compactness of the unit ball there is a convergent subnet f^λ, and we define μ to be its weak-* limit. It is then easy to verify that μ is a mean. Since

$$|\langle f^\lambda - gf^\lambda, \varphi\rangle| \leq \|f^\lambda - gf^\lambda\|_V \|\varphi\|$$

and the right hand side tends to 0, we obtain $\mu(\varphi) = \mu(g\varphi)$.

Conversely, by Goldstine's theorem, (see, e.g., [8, Theorem 2.6.26]) as $\mu \in W_0(G, X)^{**}$, μ is the weak-* limit of a bounded net of elements $f^\lambda \in W_0(G, X)$. We note that we can choose f^λ in such a way that $\pi(f^\lambda) = 1$. Indeed, given f^λ with $\pi(f^\lambda) = c_\lambda \to \mu(\pi) = 1$ we replace each f^λ by

$$f^\lambda + (1 - c_\lambda)\delta_e.$$

Since $(1 - c_\lambda)\delta_e \to 0$ in norm in $W_0(G, X)$, μ is the weak-* limit of the net $f^\lambda + (1 - c_\lambda)\delta_e$ as required.

Since μ is invariant, we have that for every $g \in G$, $gf^\lambda \to g\mu = \mu$, so that $gf^\lambda - f^\lambda \to 0$ in the weak-* topology. However, for every $g \in G$, $gf^\lambda - f^\lambda \in W_0(G, X)$, and so the convergence is in fact in the weak topology on $W_0(G, X)$.

For every λ, we regard the family $(gf^\lambda - f^\lambda)_{g\in G}$ as an element of the product $\prod_{g\in G} W_0(G, X)$, noting that this sequence converges to 0 in the Tychonoff weak topology.

Now $\prod_{g\in G} W_0(G, X)$ is a Fréchet space in the Tychonoff norm topology, so by Mazur's theorem there exists a sequence f^n of convex combinations of f^λ such that $(gf^n - f^n)_{g\in G}$ converges to zero in the Fréchet topology. Thus there exists a sequence f^n of elements of $W_0(G, X)$ such that for every $g \in G$, $\|gf^n - f^n\| \to 0$ in $W_0(G, X)$.

The result then follows from Remark 12. □

4. Equivariant means on geometric modules

Given an invariant mean $\mu \in W_0(G, X)^{**}$ for the action of G on X and an ℓ^1-geometric G-$C(X)$ module $\mathcal{E}$, we define a G-equivariant averaging operator $\mu_{\mathcal{E}} : \ell^\infty(G, \mathcal{E}^*) \to \mathcal{E}^*$ which we will also refer to as an equivariant mean for the action.

To do so, following an idea from [3], we introduce a linear functional $\sigma_{\tau,\nu}$ on $W_{00}(G,X)$. Given a Banach space $\mathcal{E}$ define $\ell^\infty(G,\mathcal{E})$ to be the space of functions $f : G \to \mathcal{E}$ such that $\sup_{g\in G} \|f(g)\|_{\mathcal{E}} < \infty$. If G acts on $\mathcal{E}$ then the action of the group G on the space $\ell^\infty(G,\mathcal{E})$ is defined in an analogous way to the action of G on V, using the induced action of G on $\mathcal{E}$:

$$(g\tau)_h = g(\tau_{g^{-1}h}),$$

for $\tau \in \ell^\infty(G,\mathcal{E})$ and $g \in G$.

Let us assume that $\mathcal{E}$ is an ℓ^1-geometric G-$C(X)$ module, and let $\tau \in \ell^\infty(G,\mathcal{E}^*)$. Choose a vector $\nu \in \mathcal{E}$ and define a linear functional $\sigma_{\tau,\nu} : W_{00}(G,X) \to \mathbb{R}$ by

$$\sigma_{\tau,\nu}(f) = \langle \sum_{h\in G} f_h \tau_h, \nu\rangle \tag{1}$$

for every $f \in W_{00}(G,X)$. If we now use Lemma 10 together with the support condition required of elements of $W_{00}(G,X)$ then we have the estimate

$$|\sigma_{\tau,\nu}(f)| \leq \left\| \sum_h f_h\tau_h \right\| \|\nu\| \leq 2\|f\|\|\tau\|\|\nu\|.$$

This estimate completes the proof of the following.

Lemma 14. *Let $\mathcal{E}$ be an ℓ^1-geometric G-$C(X)$ module. For every $\tau \in \ell^\infty(G,\mathcal{E}^*)$ and every $\nu \in \mathcal{E}$ the linear functional $\sigma_{\tau,\nu}$ on $W_{00}(G,X)$ is continuous and so it extends to a continuous linear functional on $W_0(G,X)$.*

Lemma 15. *The map $\ell^\infty(G,\mathcal{E}^*) \times \mathcal{E} \to W_0(G,X)^*$ defined by $(\tau,\nu) \mapsto \sigma_{\tau,\nu}$ is G-equivariant.*

Proof.

$$\begin{aligned}
\sigma_{g\tau,g\nu}(f) &= \left\langle \sum_h f_h g(\tau_{g^{-1}h}), g\nu \right\rangle = \left\langle g\sum_h (g^{-1}\cdot f_h)\tau_{g^{-1}h}, g\nu \right\rangle \\
&= \left\langle \sum_h (g^{-1}\cdot f_h)\tau_{g^{-1}h}, \nu \right\rangle = \left\langle \sum_h (g^{-1}f)_{g^{-1}h}\tau_{g^{-1}h}, \nu \right\rangle \\
&= \sigma_{\tau,\nu}(g^{-1}f) = (g\sigma_{\tau,\nu})(f).
\end{aligned}$$

□

Definition 16. *Let $\mathcal{E}$ be an ℓ^1-geometric G-$C(X)$ module, and let $\mu \in W_0(G,X)^{**}$ be an invariant mean for the action. We define $\mu_{\mathcal{E}} : \ell^\infty(G,\mathcal{E}^*) \to \mathcal{E}^*$ by*

$$\langle \mu_{\mathcal{E}}(\tau), \nu\rangle = \langle \mu, \sigma_{\tau,\nu}\rangle,$$

for every $\tau \in \ell^\infty(G,\mathcal{E}^*)$, and $\nu \in \mathcal{E}$.

Lemma 17. *Let $\mathcal{E}$ be an ℓ^1-geometric G-$C(X)$ module, and let $\mu \in W_0(G,X)^{**}$ be an invariant mean for the action.*

(1) *The map $\mu_{\mathcal{E}}$ defined above is G-equivariant.*

(2) *If* $\tau \in \ell^\infty(G, \mathcal{E}^*)$ *is constant then* $\mu_{\mathcal{E}}(\tau) = \tau_e$.

Proof.

$$\begin{aligned}\langle \mu_{\mathcal{E}}(g\tau), \nu\rangle &= \mu(\sigma_{g\tau,\nu}) = \mu(g \cdot \sigma_{\tau, g^{-1}\nu}) = \mu(\sigma_{\tau,g^{-1}\nu}) \\ &= \langle \mu_{\mathcal{E}}(\tau), g^{-1}\nu\rangle = \langle g \cdot (\mu_{\mathcal{E}}(\tau)), \nu\rangle.\end{aligned}$$

If τ is constant then

$$\begin{aligned}\sigma_{\tau,\nu}(f) &= \left\langle \sum_h f_h \tau_h, \nu \right\rangle = \left\langle \left(\sum_h f_h\right) \tau_e, \nu \right\rangle \\ &= \langle (\pi(f) 1_X)\tau_e, \nu\rangle = \langle \pi(f)\tau_e, \nu\rangle = \langle \tau_e, \nu\rangle \pi(f).\end{aligned}$$

So $\sigma_{\tau,\nu} = \langle \tau_e, \nu\rangle \pi$ and

$$\langle \mu_{\mathcal{E}}(\tau), \nu\rangle = \mu(\sigma_{\tau,\nu}) = \mu(\langle \tau_e, \nu\rangle \pi) = \langle \tau_e, \nu\rangle,$$

hence $\mu_{\mathcal{E}}(\tau) = \tau_e$. □

5. AMENABLE ACTIONS AND BOUNDED COHOMOLOGY

Let $\mathcal{E}$ be a Banach space equipped with an isometric action by G. Then we consider a cochain complex $C_b^m(G, \mathcal{E}^*)$ which in degree m consists of G-equivariant bounded cochains $\phi : G^{m+1} \to \mathcal{E}^*$ with values in the Banach dual $\mathcal{E}^*$ of $\mathcal{E}$ which is equipped with the natural differential d as in the homogeneous bar resolution. Bounded cohomology with coefficients in $\mathcal{E}^*$ will be denoted by $H_b^*(G, \mathcal{E}^*)$.

Definition 18. *Let* G *be a countable discrete group acting by homeomorphisms on a compact Hausdorff topological space* X. *The function*

$$J(g_0, g_1) = \delta_{g_1} - \delta_{g_0}$$

is a bounded cochain of degree 1 *with values in* $N_{00}(G, X)$, *and in fact it is a bounded cocycle and so represents a class in* $H_b^1(G, N_0(G, X)^{**}$, *where we regard* $N_{00}(G, X)$ *as a subspace of* $N_0(G, X)^{**}$. *We call* $[J]$ *the Johnson class of the action.*

Theorem B. *Let* G *be a countable discrete group acting by homeomorphisms on a compact Hausdorff topological space* X. *Then the following are equivalent*

(1) *The action of* G *on* X *is topologically amenable.*

(2) *The class* $[J] \in H_b^1(G, N_0(G, X)^{**})$ *is trivial.*

(3) $H_b^p(G, \mathcal{E}^*) = 0$ *for* $p \geq 1$ *and every* ℓ^1*-geometric* G-$C(X)$ *module* $\mathcal{E}$.

Proof. We first show that (1) is equivalent to (2). The short exact sequence of G-modules

$$0 \to N_0(G, X) \to W_0(G, X) \xrightarrow{\pi} \mathbb{R} \to 0$$

leads, by taking double duals, to the short exact sequence

$$0 \to N_0(G,X)^{**} \to W_0(G,X)^{**} \to \mathbb{R} \to 0$$

which in turn gives rise to a long exact sequence in bounded cohomology

$$H^0_b(G, N_0(G,X)^{**}) \to H^0_b(G, W_0(G,X)^{**}) \to H^0_b(G,\mathbb{R}) \to H^1_b(G, N_0(G,X)^{**}) \to \dots$$

The Johnson class $[J]$ is the image of the class $[1] \in H^0_b(G,\mathbb{R})$ under the connecting homomorphism $d : H^0_b(G,\mathbb{R}) \to H^1(G, N_0(G,X)^{**})$, and so $[J] = 0$ if and only if $d[1] = 0$. By exactness of the cohomology sequence, this is equivalent to $[1] \in \mathrm{Im}\,\pi^{**}$, where $\pi^{**} : H^0_b(G, W_0(G,X)^{**}) \to H^0_b(G,\mathbb{R})$ is the map on cohomology induced by the summation map π. Since $H^0_b(G, W_0(G,X)^{**}) = (W_0(G,X)^{**})^G$ and $H^0_b(G,\mathbb{R}) = \mathbb{R}$ we have that $[J] = 0$ if and only if there exists an element $\mu \in W_0(G,X)^{**}$ such that $\mu = g\mu$ and $\mu(\pi) = 1$. Thus μ is an invariant mean for the action and the equivalence with amenability of the action follows from Theorem A.

We turn to the implication (1) implies (3). Since G acts amenably on X there is, by Theorem A, an invariant mean μ associated with the action. For every $h \in G$ and for every equivariant bounded cochain ϕ we define $s_h\phi : G^p \to \mathcal{E}^*$ by $s_h\phi(g_0,\dots,g_{p-1}) = \phi(g, g_0,\dots,g_{p-1})$; we note that for fixed h, $s_h\phi$ is not equivariant in general. However, the map s_h does satisfy the identity $ds_h + s_h d = 1$ for every $h \in G$, and we will now construct an equivariant contracting homotopy, adapting an averaging procedure introduced in [3].

For $\phi \in C^p_b(G,\mathcal{E}^*)$ let $\widehat{\phi} : G^p \to \ell^\infty(G,\mathcal{E}^*)$ be defined by $\widehat{\phi}(\mathbf{g})(h) = s_h\phi(\mathbf{g})$, for $\mathbf{g} = (g_0,\dots g_{p-1})$.

Note that for every $k, h \in G$,

$$\begin{aligned}
\widehat{\phi}(kg_0,\dots,kg_{p-1})(h) &= \phi(h, kg_0,\dots,kg_{p-1}) = k(\phi(k^{-1}h, g_0,\dots,g_{p-1}))\\
&= k(\widehat{\phi}(g_0,\dots,g_{p-1})(k^{-1}h))\\
&= (k(\widehat{\phi}(g_0,\dots,g_{p-1})))(h)
\end{aligned}$$

so $\widehat{\phi}(k\mathbf{g}) = k(\widehat{\phi}(\mathbf{g}))$.

We can now define a map $s : C^p(G,\mathcal{E}^*) \to C^{p-1}(G,\mathcal{E}^*)$:

$$s\phi(\mathbf{g}) = \mu_{\mathcal{E}}(\widehat{\phi}(\mathbf{g})),$$

where $\mu_{\mathcal{E}} : \ell^\infty(G,\mathcal{E}^*) \to \mathcal{E}^*$ is the map defined in Lemma 17 using the invariant mean μ. Note that $\|\mu_{\mathcal{E}}\| \le 2\|\mu\|$, and $\|\widehat{\phi}(\mathbf{g})\| \le \sup\{\|\phi(\mathbf{k})\| \mid \mathbf{k} \in G^{p+1}\}$. Hence $s\phi$ is bounded.

For every cochain ϕ, $k(s\phi) = s(k\phi) = s\phi$ since $\widehat{\phi}$ and $\mu_{\mathcal{E}}$ are equivariant.

The map s provides a contracting homotopy for the complex $C^*_b(G,\mathcal{E}^*)$ which can be seen as follows. As $\mu_{\mathcal{E}} : \ell^\infty(G,\mathcal{E}^*) \to \mathcal{E}^*$ is a linear operator it follows that for a given $\phi \in C^p_b(G,\mathcal{E}^*)$, and a $p+1$-tuple of arguments $\mathbf{k} = (k_0,\dots,k_p)$, $ds\phi$ is obtained by applying the mean $\mu_{\mathcal{E}}$ to the map $g \mapsto ds_g\phi(\mathbf{k})$, while $sd\phi$ is obtained by applying $\mu_{\mathcal{E}}$ to the function $g \mapsto s_g d\phi(\mathbf{k})$. Thus

$$(sd + ds)\phi(\mathbf{k}) = \mu_{\mathcal{E}}(g \mapsto (ds_g + s_g d)\phi(\mathbf{k})).$$

Given that $ds_g + s_g d = 1$ for every $g \in G$, for every $\mathbf{g} \in G^{p+1}$ the function $g \mapsto (ds_g + s_g d)\phi(\mathbf{k}) = \phi(\mathbf{k}) \in \mathcal{E}^*$ is constant, and so by Lemma 17,

$$(sd + ds)\phi(\mathbf{k}) = (ds_e + s_e d)\phi(\mathbf{k}) = \phi(\mathbf{k}).$$

Thus $sd + ds = 1$, as required.

Collecting these results together, we have proved that (1) implies (3).

The fact that (3) implies (2), follows from the fact that $N_0(G, X)^*$ is an ℓ^1-geometric G-$C(X)$-module, proved in Lemma 7.

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REFERENCES

[1] C. Anantharaman-Delaroche, J. Renault, Amenable groupoids, Monographies de L'Enseignement Mathématique, 36. L'Enseignement Mathématique, Geneva, 2000.
[2] J. Block and S. Weinberger, Aperiodic tilings, positive scalar curvature and amenability of spaces. J. Amer. Math. Soc. 5 (1992), no. 4, 907–918.
[3] J. Brodzki, G. A. Niblo and N. J. Wright, A cohomological characterisation of Yu's Property A for metric spaces. Preprint, University of Southampton ePrints November 2009: arXiv:1002.5040.
[4] N. P. Brown, N. Ozawa, C^*-algebras, and finite dimensional approximations, Graduate Studies in Mathematics, 88, AMS, Rhode Island 2008.
[5] R. G. Douglas and P. W. Nowak, Invariant expectations and vanishing of bounded cohomology for exact groups. Preprint, arXiv:1001.0718v3.
[6] B. E. Johnson, Cohomology of Banach Algebras, Memoirs of the AMS Number 127, 1972, AMS, Providence, Rhode Island.
[7] N. Higson, J. Roe, Amenable group actions and the Novikov conjecture, J. Reine Angew. Math. 519 (2000), 143–153.
[8] R. E. Megginson, An Introduction to Banach Space Theory, (Graduate Texts in Mathematics 183), Springer Verlag New York, 1998.
[9] N. Ozawa, Amenable actions and exactness for discrete groups, C.R. Acad. Sci. Paris Ser. I Math 330 (2000), 691-695.
[10] G. Yu, The Coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space, Inventiones Math. **139** (2000), 201–240.

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